Quiz 3 Solutions

1. Decide with justification whether

$$\int_0^1 \frac{\sec x}{x\sqrt{x}}\,\mathrm{d}x.$$

converges (Hint: do not try to compute an antiderivative for the integrand).

Solution: We have $\dfrac{\sec x}{x\sqrt{x}} \geq \dfrac{1}{x\sqrt{x}} = \dfrac{1}{x^{3/2}} > 0$. By lecture $\displaystyle\int_0^1 \frac{1}{x^{3/2}}\,\mathrm{d}x$ diverges since $\dfrac{3}{2} > 1$, hence $\displaystyle\int_0^1 \frac{\sec x}{x\sqrt{x}}\,\mathrm{d}x$ also diverges by the comparison test.

2. Find the length of the curve given by $y = \ln(\cos x)$ where $0 \leq x \leq \dfrac{\pi}{4}$.
(You may use $\displaystyle\int \sec x\,\mathrm{d}x = \ln(\sec x + \tan x) + C$ without proof)

Solution: The length is given by

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\,\mathrm{d}x \\
&\overset{\sec x > 0 \text{ on } [0, \frac{\pi}{4}]}{=} \int_0^{\frac{\pi}{4}} \sec x\,\mathrm{d}x \\
&= \ln(\sec x + \tan x)\Big|_0^{\frac{\pi}{4}} \\
&= \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1).
\end{aligned}$$

3. Find T_4 for the function $f(x) = x^2 - 1$ on the interval $[1, 3]$ (i.e. approximate the integral $\displaystyle\int_1^3 x^2 - 1\,\mathrm{d}x$ using the trapezoid rule with $n = 4$ subintervals).

Solution: $x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$ and $\Delta x = \frac{1}{2}$. Then

$$\begin{aligned} T_4 &= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) \\ &= \frac{1}{4}((1-1) + 2 \cdot (\frac{9}{4} - 1) + 2 \cdot (4-1) + 2 \cdot (\frac{25}{4} - 1) + (9-1)) \\ &= \frac{1}{4}(\frac{10}{4} + 6 + \frac{42}{4} + 8) \\ &= \frac{1}{4}(27) = \frac{27}{4} \end{aligned}$$